

Equations of Motion in Double Field Theory: From Particles to Scale Factors

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In double field theory, the equation of motion for a point particle in the background field is considered. We find that the motion is described by a geodesic flow in the doubled geometry. Inspired by the analysis on the particle motion, we consider a modified model of quantum string cosmology, which includes two scale factors.

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I. INTRODUCTION

T-duality is an important symmetry in string theory [1]. This symmetry can be interpreted as an invariance under the interchanging of coordinates and dual coordinates. Dual coordinates are also reported to be important in string field theory (SFT) [2].

Recently, Hull, Zwiebach, and Hohm constructed the theory of the massless field with a higher symmetry of spacetime including dual coordinates; This is referred to as double field theory (DFT). The original work is given in [3], and later developments can be found in [4–9].¹ Through this theory, Hull, Zwiebach, and Hohm clarified the T-duality symmetry of the massless field, new algebra, and the symmetry related to the theory.

Because T-duality was discovered and interpreted as an exchanging symmetry of the Kaluza-Klein (KK) modes and winding modes in compact space, we are simply interested in the extent to which the theory of zero mode field alone can describe the special features of string theory. More recently Jeon, Lee, and Park proposed the construction of the Yang-Mills theory as DFT [10] using their projection-compatible differential geometrical methods in [6]. It is interesting to study the duality-symmetric extension of various field theories.

In the present paper, the formalism in [6], in which the geometrical aspect of DFT is introduced is concisely reviewed, and subsequently the couplings of zero-mode fields and a particle are examined along with the equation of motion for the particle in the background fields.

One of our motivations for this investigation is the importance of clarifying the meaning of geodesics in space with consideration of dual coordinates. In addition, since the origin of the generalized metric is considered for a string, we believe that the coupling to the background fields can be neatly interpreted for particles as well.

In the next section, we review the projection-compatible approach to DFT [6]. In Sec. III, the equation of motion for a string, as proposed by Duff [12] is reviewed. After these reviews, we attempt to find the geometrical meaning of particle motion in Sec. IV. In Sec. V, we analyze the particle motion in the case of Hamiltonian formalism. The equation of geodesic flow is applied to the KK type model in Sec. VI. The quantization of particles in the DFT background is studied in Sec. VII. The KK mass spectrum of the scalar field is also investigated. Inspired by the above approach, we propose a modified model of DFT for

¹ Other developments that we were not aware of during the completion of the main crux of this study are given in [19–23].

cosmological evolution equations in Sec. VIII. In Sec. IX, the mini-superspace formulation of our model is explained, and the quantum cosmological aspects of our model are examined in Sec. X. The final section is devoted to the summary and prospects.

II. REVIEW OF PROJECTION-COMPATIBLE APPROACH

The constant metric is assumed to be expressed as the following $2D \times 2D$ matrix,

$$\eta_{AB} = \begin{pmatrix} 0 & \delta^a{}_\nu \\ \delta_\mu{}^b & 0 \end{pmatrix}. \quad (2.1)$$

Here, the suffixes A, B, \dots range over $1, 2, \dots, 2D$, while μ, ν, \dots as well as a, b, \dots range over $1, 2, \dots, D$. The suffixes are entirely raised and lowered by this constant metric. Of course,

$$\eta^{AC} \eta_{CB} = \delta^A{}_B, \quad (2.2)$$

is satisfied.

The generalized metric is defined as follows.

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{ab} & -g^{a\sigma} b_{\sigma\nu} \\ b_{\mu\sigma} g^{\sigma b} & g_{\mu\nu} - b_{\mu\rho} g^{\rho\sigma} b_{\sigma\nu} \end{pmatrix}. \quad (2.3)$$

Here, $g_{\mu\nu}$ and $b_{\mu\nu}$ are the metric in D dimensions and the antisymmetric tensor, respectively. It should be noted that the inverse of the generalized metric, or the matrix \mathcal{H}^{AB} satisfying

$$\mathcal{H}^{AC} \mathcal{H}_{CB} = \delta^A{}_B, \quad (2.4)$$

is obtained by raising the suffixes by the constant metric, as

$$\mathcal{H}^{AB} = \eta^{AC} \mathcal{H}_{CD} \eta^{DB} = \begin{pmatrix} g_{ab} - b_{a\rho} g^{\rho\sigma} b_{\sigma b} & b_{a\sigma} g^{\sigma\nu} \\ -g^{\mu\sigma} b_{\sigma b} & g^{\mu\nu} \end{pmatrix}. \quad (2.5)$$

The following projection matrices are defined on the basis of the existence of two kinds of metrics.

$$P^A{}_B \equiv \frac{1}{2}(\delta^A{}_B + \mathcal{H}^A{}_B), \quad \bar{P}^A{}_B \equiv \frac{1}{2}(\delta^A{}_B - \mathcal{H}^A{}_B), \quad (2.6)$$

which satisfy

$$P^2 = P, \quad \bar{P}^2 = \bar{P}, \quad P\bar{P} = \bar{P}P = 0. \quad (2.7)$$

From this, one can derive the identities

$$P(\partial_A P)P = \bar{P}(\partial_A \bar{P})\bar{P} = 0 \quad \text{or} \quad P_D{}^B(\partial_A \mathcal{H}_{BC})P^C{}_E = \bar{P}_D{}^B(\partial_A \mathcal{H}_{BC})\bar{P}^C{}_E = 0. \quad (2.8)$$

Now, the projection-compatible derivative is defined. In other words, both the metrics are “covariantly constant,” i.e.,

$$\nabla_A \eta_{BC} = 0, \quad \nabla_A \mathcal{H}_{BC} = 0. \quad (2.9)$$

So, the covariant derivative of the projection of an arbitrary tensor coincides with the projection of the covariant derivative of the tensor. Jeon et al.[6] found that the covariant derivatives including the following connection have the character,

$$\Gamma_{ABC} \equiv 2P_{[A}{}^D \bar{P}_{B]}{}^E \partial_C P_{DE} + 2(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E) \partial_D P_{EC}. \quad (2.10)$$

They also obtained the action for the generalized metric, which was previously found by Hohm, Hull, and Zwiebach [3–5, 7–9]

$$S = \int dx d\tilde{x} e^{-2d} \left(\frac{1}{8} \mathcal{H}^{AB} \partial_A \mathcal{H}^{CD} \partial_B \mathcal{H}_{CD} - \frac{1}{2} \mathcal{H}^{AB} \partial_B \mathcal{H}^{CD} \partial_D \mathcal{H}_{AC} - 2 \partial_A d \partial_B \mathcal{H}^{AB} + 4 \mathcal{H}^{AB} \partial_A d \partial_B d \right), \quad (2.11)$$

from the consideration of the projection-compatible geometrical quantities. Here, $e^{-2d} = \sqrt{-g} e^{-2\phi}$ and ϕ is the dilaton field. If we set all the derivatives on the fields with respect to the dual coordinate zero ($\tilde{\partial}^a = 0$), the action for the well-known effective theory for the zero-mode (massless) field in string theory is obtained as

$$S = \int dx \sqrt{-g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right], \quad (2.12)$$

where the three-form field $H = db$ is the field strength of the Kalb-Ramond 2-form b_{ij} .

In a cosmological context, we consider that all the fields depend only on time coordinate t and assume that the metric and the antisymmetric tensor in $D + 1$ dimensions are respectively,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & G_{\mu\nu}(t) \end{pmatrix}, \quad b_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & B_{\mu\nu}(t) \end{pmatrix}. \quad (2.13)$$

Then, the action given in (2.11) is reduced to

$$S \rightarrow - \int dt e^{-\Phi} \left(\frac{1}{8} \text{Tr} (\dot{M} \eta \dot{M} \eta) + \dot{\Phi}^2 \right), \quad (2.14)$$

where the dot represents the time derivative, $\Phi \equiv 2d$, and

$$M \equiv \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}. \quad (2.15)$$

It is also assumed that η is a $2D \times 2D$ matrix. The action exactly corresponds to the one considered in the string cosmology and which has been studied by several authors [11].

III. THE EQUATION OF MOTION FOR A STRING

In this section, let us review the work of Duff [12]. He investigated the equation of motion and the duality in string theory. In the next section, we will study the equation of motion of a particle in DFT on the basis of the consideration in the present section. Although the extent of the remaining influences of the string duality to the motion of particles is known, the consideration on strings and particles in the background fields is indispensable because “the geometry of spacetime tells particles how to move.”² After all, we are interested in various geometrical characters in DFT.

Now, let us turn to the review of the paper of Duff [12]. Coordinates are combined with dual coordinates to be

$$X^A \equiv \begin{pmatrix} \tilde{x}_a \\ x^\mu \end{pmatrix}, \quad (3.1)$$

which is an $O(D, D)$ vector.

For simplicity we consider the standard line element on the world sheet, $\eta_{ij}d\xi^i d\xi^j = d\tau^2 - d\sigma^2$. At this time, Duff’s equation of motion is [12]

$$\partial^i(\mathcal{H}_{AB}\partial_i X^B) = 0, \quad (3.2)$$

where the notation is renewed.

The equation of duality, or the “BPS” equation providing the solution of this equation of motion is given by [12]

$$\eta_{AB}X'^B = \mathcal{H}_{AB}\dot{X}^B, \quad (3.3)$$

where the prime ($'$) and the dot ($\dot{}$) denote the derivatives with respect to σ and τ , respectively.

² The original sentence is found in [13].

This equation is equivalent to

$$P_{AB} \frac{\partial X^B}{\partial \sigma^-} = 0, \quad \bar{P}_{AB} \frac{\partial X^B}{\partial \sigma^+} = 0, \quad (3.4)$$

where $\sigma^\pm = \tau \pm \sigma$. Then, the equation can be expressed in terms of $X_L^B(\sigma^+)$ and $X_R^B(\sigma^-)$, each of which is the function of their respective coordinates, as

$$P^A{}_B X^B = X_L^A, \quad \bar{P}^A{}_B X^B = X_R^A, \quad (3.5)$$

in the fixed background fields. As is well known, the string coordinate and its dual are written in $X = X_L + X_R$ and $\tilde{X} = X_L - X_R$, respectively, and hence, the relation of $X \sim \tilde{X}$ will show that the spacetime described by the set of coordinates is the projected space obtained with the projection matrix P . This is consistent with the projection-compatible procedure of Jeon et al.

The relation between the duality and the projection in the equation of the string has been understood above. Duff explicitly claimed in his paper [12] that the duality (rotation) is not the symmetry of the action but the symmetry of the equation of motion of the string. In other words, the equation of motion does not necessarily originate from an action with duality invariance. This certainly appears to be the case; the equation of motion (3.2) does not have the action in most general background fields.

IV. IS THE EQUATION OF MOTION FOR A PARTICLE THE GEODESIC EQUATION?

Next, we consider the equation of motion for a particle. Let us start with the context of the differential geometry with projection. For a usual common aspect of general relativity, the geodesic equation is given by the following expression [14]

$$U^\mu \nabla_\mu U^\nu = 0, \quad (4.1)$$

where $U^\mu \equiv \frac{dx^\mu}{ds} = \dot{x}^\mu$, s being a parameter.

The corresponding equation in the projection compatible geometry of Jeon et al. is considered to be

$$U^A \nabla_A U^B = U^A (\partial_A U^B + \Gamma_A{}^B{}_C U^C) = 0, \quad (4.2)$$

where $U^A = (\tilde{U}_a, U^\mu)^T = \frac{dX^A}{ds}$.

We explicitly calculate the connection $\Gamma_A{}^B{}_C$ for $b_{\mu\nu} = 0$. The suffixes μ and ν indicate those of the usual coordinates while a and b are those of dual coordinates. Note that both suffixes can stick to the single metric g . Either letter ranges over 1 to D , which is considered carefully in the sum by the Einstein rule. Moreover, the derivative with respect to the dual coordinates is denoted by $\tilde{\partial}$. The elements of $\Gamma_A{}^B{}_C$ are found to be

$$\Gamma_\nu{}^\mu{}_\lambda = \{\nu{}^\mu{}_\lambda\} - \frac{1}{2}g_{\lambda\sigma}\partial_\nu g^{\sigma\mu}, \quad (4.3)$$

$$\Gamma_\nu{}^{\mu c} = \frac{1}{2}\left(g^{\mu\sigma}\tilde{\partial}^c g_{\sigma\nu} - g^{cd}\tilde{\partial}^\mu g_{d\nu}\right), \quad (4.4)$$

$$\Gamma^{a\mu}{}_\lambda = -\{a{}^\mu{}_\lambda\} + \frac{1}{2}g^{\mu\sigma}\tilde{\partial}^a g_{\sigma\lambda}, \quad (4.5)$$

$$\Gamma^{a\mu b} = \frac{1}{2}\left(g^{b\sigma}\partial_\sigma g^{\mu a} - g^{\mu\sigma}\partial_\sigma g^{ab}\right), \quad (4.6)$$

$$\Gamma^b{}_a{}^c = \{b{}_a{}^c\} - \frac{1}{2}g^{cd}\tilde{\partial}^b g_{da}, \quad (4.7)$$

$$\Gamma^b{}_{a\lambda} = \frac{1}{2}\left(g_{ad}\partial_\lambda g^{db} - g_{\lambda d}\partial_a g^{db}\right), \quad (4.8)$$

$$\Gamma_{\nu a}{}^b = -\{\nu{}^b{}_a\} + \frac{1}{2}g_{ad}\partial_\nu g^{db}, \quad (4.9)$$

$$\Gamma_{\nu a\lambda} = \frac{1}{2}\left(g_{\lambda b}\tilde{\partial}^b g_{a\nu} - g_{ab}\tilde{\partial}^b g_{\nu\lambda}\right), \quad (4.10)$$

where

$$\{\nu{}^\mu{}_\lambda\} \equiv \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda}), \quad (4.11)$$

$$\{b{}_a{}^c\} \equiv \frac{1}{2}g_{ad}(\tilde{\partial}^b g^{dc} + \tilde{\partial}^c g^{db} - \tilde{\partial}^d g^{bc}). \quad (4.12)$$

From the project space ansatz $\bar{P}U = 0$, we are forced to use $\tilde{U}_a = g_{a\nu}U^\nu$. Moreover, we set $\tilde{\partial}g = 0$ as in the interpretation of DFT. Next, we find that the equation (4.2) reads

$$U^\mu\partial_\mu U^\nu + \frac{1}{2}g^{\nu\mu}(\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\mu\rho})U^\rho U^\sigma = 0. \quad (4.13)$$

This equation is equivalent to $\frac{d}{ds}(g_{\mu\sigma}\dot{x}^\sigma)$, which is similar to the equation of motion for the string considered by Duff. However, it is obviously different from the usual geodesic equation in general relativity (or differential geometry). It seems that the physical interpretation of the equation is difficult. If the equation is true, the Newtonian gravity cannot be obtained by considering the ‘‘Newtonian limit.’’

In general, it is understood that the usage of the projection has a problem. From brief calculation, the following equation can be shown,

$$P_D{}^A\Gamma_{(A}{}^B{}_{C)}P^C{}_E = P_D{}^A\left[\frac{1}{2}\mathcal{H}^{BF}(\partial_A\mathcal{H}_{FC} + \partial_C\mathcal{H}_{FA})\right]P^C{}_E. \quad (4.14)$$

It is interesting that because $P\partial_A\mathcal{H}P=0$, we can show that

$$P_D{}^A\Gamma_{(A}{}^B{}_{C)}P^C{}_E=P_D{}^A\left[\frac{1}{2}\mathcal{H}^{BF}(\partial_A\mathcal{H}_{FC}+\partial_C\mathcal{H}_{FA}-\partial_F\mathcal{H}_{AC})\right]P^C{}_E \quad (4.15)$$

is also true. In this case, the equation is considered to be the one derived from the Lagrangian

$$L=\frac{1}{2}\mathcal{H}_{AB}\dot{X}^A\dot{X}^B, \quad (4.16)$$

with the projection such that the condition $\bar{P}U=0$ is imposed.

No physical interpretation was obtained from (4.2). Since there was obviously no physical requirement in the starting point, we can correct the equation for a particle. However, a clear geometrical point of view is missing.

Although the projection-compatible method is still valid for the field theory, we noticed that the projection of the motion of the particle (or string) should be handled as a constraint. In the next section, we consider the motion of a particle in the spacetime described by the generalized metric in the analytical mechanics of the constraint system.

V. PROJECTION AND GEODESIC FLOW

We treat the constraint using the Lagrange multiplier. The following Lagrangian is adopted, and the mechanics derived from it are considered.

$$L=\frac{1}{2}\mathcal{H}_{AB}\dot{X}^A\dot{X}^B+\lambda^A\bar{P}_{AB}\dot{X}^B. \quad (5.1)$$

Here, λ^A is an undecided multiplier. The Euler-Lagrange equation leads to the constraint $\bar{P}\dot{X}=0$, which is adopted as the projection in the previous section.

We find that the conjugate momentum of X^A is

$$p_A=\frac{\partial L}{\partial \dot{X}^A}=\mathcal{H}_{AB}\dot{X}^B+\lambda^B\bar{P}_{BA}, \quad (5.2)$$

and the conjugate momentum of λ^A is

$$\pi_A=\frac{\partial L}{\partial \dot{\lambda}^A}=0. \quad (5.3)$$

Then, the Hamiltonian is defined as

$$H=p_A\dot{X}^A+\pi_A\dot{\lambda}^A-L=\frac{1}{2}\mathcal{H}^{AB}(p_A-\lambda^C\bar{P}_{CA})(p_B-\lambda^D\bar{P}_{DB}). \quad (5.4)$$

The equation for the conjugate momentum of the multiplier λ^A becomes

$$\dot{\pi}_A = -\frac{\partial H}{\partial \lambda^A} = \bar{P}_{AB} \mathcal{H}^{BC} (p_C - \lambda^D \bar{P}_{DC}) = -\bar{P}_{AB} (p^B - \lambda^B), \quad (5.5)$$

which will vanish. The multiplier can be determined from the above equation as

$$\lambda_A = p_A + P_{AB} M^B, \quad (5.6)$$

where M^B is an arbitrary vector. When this solution is substituted into the Hamiltonian (5.4), we obtain a new Hamiltonian

$$H_\star = \frac{1}{2} \mathcal{H}^{AB} P_{AC} p^C P_{BD} p^D = \frac{1}{2} P^{AB} p_A p_B, \quad (5.7)$$

where the arbitrariness in the solution has disappeared. If we take $p_A = (\tilde{p}^a, p_\mu)$, the new Hamiltonian can be rewritten as

$$H_\star = \frac{1}{2} \tilde{p}^a p_a + \frac{1}{4} g^{\mu\nu} (p_\mu - b_{\mu\rho} \tilde{p}^\rho) (p_\nu - b_{\nu\sigma} \tilde{p}^\sigma) + \frac{1}{4} g_{ab} \tilde{p}^a \tilde{p}^b. \quad (5.8)$$

Using the new Hamiltonian, we obtain

$$\dot{X}^A = \frac{\partial H_\star}{\partial p_A} = P^{AB} p_B. \quad (5.9)$$

This clearly satisfies the constraint $\bar{P} \dot{X} = 0$. Furthermore, we can obtain the equation of motion

$$\dot{p}_A = -\frac{\partial H_\star}{\partial X^A} = -\frac{1}{2} \partial_A P^{BC} p_B p_C = -\frac{1}{4} \partial_A \mathcal{H}^{BC} p_B p_C. \quad (5.10)$$

These equations describe the geodesic flow in the system. Since equation (5.9) is not always solvable for p_A , rewriting it a generalized second-rank differential equation for X^A is not possible. That is, the expression for the second derivative of X^A becomes

$$\begin{aligned} \ddot{X}^A &= \frac{d}{ds} (P^{AB} p_B) = \dot{X}^C (\partial_C P^{AB}) p_B + P^{AB} \dot{p}_B \\ &= \dot{X}^C (\partial_C P^{AB}) p_B - \frac{1}{2} P^{AB} \partial_B P^{CD} p_C p_D, \end{aligned} \quad (5.11)$$

where the momentum remains unsolved in general.

Now, let us take the condition $\tilde{\partial}^a = 0$ for the correspondence with DFT. In this case, the D -dimensional fixed vector $\tilde{p}^a = \tilde{p}_0^a$ (remembering that $p_A = (\tilde{p}^a, p_\mu)$) is a general solution, because $\dot{\tilde{p}}^a = 0$ can be obtained from equation (5.10).

In this section, we consider $\tilde{p}_0^a = 0$ (for all dual coordinate components) to assess the interpretation of the equation. In this case, the equation (5.9) leads to $\dot{x}^\mu = \frac{1}{2}g^{\mu\nu}p_\nu$. Namely, $p_\mu = 2g_{\mu\nu}\dot{x}^\nu$ is obtained, and we substitute it into (5.11) and obtain

$$\begin{aligned}\ddot{x}^\mu &= \dot{x}^\nu(\partial_\nu g^{\mu\rho})g_{\rho\sigma}\dot{x}^\sigma - \frac{1}{2}g^{\mu\rho}(\partial_\rho g^{\sigma\nu})g_{\sigma\lambda}\dot{x}^\lambda g_{\nu\tau}\dot{x}^\tau \\ &= -\frac{1}{2}g^{\mu\nu}(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma})\dot{x}^\rho\dot{x}^\sigma.\end{aligned}\quad (5.12)$$

This equation is nothing but $\ddot{x}^\mu + \left\{\begin{smallmatrix}\mu \\ \rho\sigma\end{smallmatrix}\right\}\dot{x}^\rho\dot{x}^\sigma = 0$, and the geodesic equation in a usual D dimensional spacetime. It should be noted that the the geodesic equation has been obtained under the condition $\tilde{p}_a = 0$, with and without the existence of $b_{\mu\nu}$.

We have obtained the geodesic equation in the D -dimensional spacetime from the geodesic flow in the $2D$ -dimensional space described by the generalized metric with natural assumptions. In the next section, we will show the dynamical equation with non-zero \tilde{p}_0^a .

VI. KALUZA-KLEIN BACKGROUND FIELDS

In this section, we assume the factorized background fields, namely, the KK background fields. Because the signature definition of the D -dimensional metric is not essential here, it is assumed $\mu = 1, 2, 3, \dots, D$. We consider the KK independent of the *first* coordinate, i.e., $\partial_1 = 0$. From equation (5.10), $p_1 = \text{constant}$ becomes the solution in this case. Moreover, as in the previous section, $\tilde{\partial}^a = 0$ is assumed and, therefore, \tilde{p}^a is a constant vector. Here, we suppose that $\tilde{p}^i = 0$ ($i = 2, 3, \dots, D$) while $\tilde{p}^1 = \text{constant}$.

Taking all the assumptions into account, we rewrite the Hamiltonian as

$$\begin{aligned}H_\star(\{x^i, p_i\}) &= \frac{1}{2}\tilde{p}^1 p_1 + \frac{1}{4}g^{ij}(p_i - \tilde{p}^1 B_i)(p_j - \tilde{p}^1 B_j) \\ &\quad + \frac{1}{2}g^{i1}p_1(p_i - \tilde{p}^1 B_i) + \frac{1}{4}g^{11}p_1^2 + \frac{1}{4}g_{11}(\tilde{p}^1)^2,\end{aligned}\quad (6.1)$$

where $B_i \equiv b_{i1}$.

In addition, we now consider the KK ansatz; for this, the D -dimensional metric can be decomposed as

$$g_{\mu\nu} = \begin{pmatrix} R^2 & R^2 A_i \\ R^2 A_j & \hat{g}_{ij} + R^2 A_i A_j \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} R^{-2} + A_k A_\ell \hat{g}^{k\ell} & -A_\ell \hat{g}^{i\ell} \\ -A_k \hat{g}^{kj} & \hat{g}^{ij} \end{pmatrix}, \quad (6.2)$$

where R (constant) is interpreted as the length scale of the *first* dimension.

From all the statements above, we finally obtain

$$H_\star(\{x^i, p_i\}) = \frac{1}{4} \hat{g}^{ij} (p_i - p_1 A_i - \tilde{p}^1 B_i) (p_j - p_1 A_j - \tilde{p}^1 B_j) + \frac{1}{4} \left(\frac{1}{R} p_1 + R \tilde{p}^1 \right)^2. \quad (6.3)$$

At this time, we immediately obtain

$$\dot{x}^i = \frac{\partial H_\star}{\partial p_i} = \frac{1}{2} \hat{g}^{ij} (p_j - p_1 A_j - \tilde{p}^1 B_j). \quad (6.4)$$

Substituting this into $\dot{p}_i = -\frac{\partial H_\star}{\partial x^i}$ and eliminating p_i and their derivatives, we have the equation of motion

$$\ddot{x}^i + \hat{\Gamma}_{jk}^i \dot{x}^j \dot{x}^k = \hat{g}^{ij} \mathcal{F}_{jk} \dot{x}^k, \quad (6.5)$$

where $\hat{\Gamma}_{jk}^i \equiv \frac{1}{2} \hat{g}^{i\ell} (\partial_j \hat{g}_{\ell k} + \partial_k \hat{g}_{\ell j} - \partial_\ell \hat{g}_{jk})$ and $\mathcal{F}_{jk} = \partial_j \frac{1}{2} (p_1 A + \tilde{p}^1 B)_k - \partial_k \frac{1}{2} (p_1 A + \tilde{p}^1 B)_j$.

It can be understood that two $U(1)$ gauge fields have a different charge coupled to the particle in the KK background field. From the classical theory, the quantization of the coupling constant cannot be decided.

We have found the equation of motion for the doubly charged particle in the constraint system in the doubled spacetime. Incidentally, the equation of motion for the particle limit of the string has also been considered, by Jafarizadeh and Rezaei-Aghdam [15].

VII. QUANTIZATION OF PARTICLES

Here, we consider the problem of the quantization of the particle. One of the effective approaches to quantization is the method of phase space action [16]. Let us consider the following action

$$I = \int ds \left[p_A \dot{X}^A - \frac{N}{2} \left(P^{AB} p_A p_B + \frac{1}{2} m^2 \right) \right], \quad (7.1)$$

where the dot indicates the derivative with respect to s . N is a Lagrange multiplier. The constant m is allowed under the symmetry and is stated below. If we consider the variation with respect to p_A , we obtain $\frac{1}{N} \frac{dX^A}{ds} = P^{AB} p_B$, whereas if we consider the variation with respect to X^A , we obtain $\frac{1}{N} \frac{dp^A}{ds} + \frac{1}{2} \partial_A P^{BC} p_B p_C = 0$. This action is invariant if the parameter s is changed, provided that N cancels the variation (that is, N acts as an einbein). Therefore, the Euler-Lagrange equations become equivalent to the expressions for the equation of motion in the previous section if $N = 1$ is assumed as a gauge choice.

When we formulate a Hamiltonian again from this phase space Lagrangian, we find

$$H_b = \frac{N}{2} \left(P^{AB} p_A p_B + \frac{1}{2} m^2 \right). \quad (7.2)$$

From the condition $\frac{\partial H}{\partial N} \approx 0$, we obtain $H_b|_{N=1} \approx 0$. If the quantum state satisfies this assumption, the constraint condition is expressed as

$$\left(P^{AB}p_Ap_B + \frac{1}{2}m^2\right)\varphi = 0, \quad (7.3)$$

where φ can be recognized as a scalar wave function. Then, the wave equation of Klein-Gordon type is obtained, if we replace $p_A \rightarrow -i\partial_A$ here with an arbitrary ordering of the operator. We can estimate the second-quantized action of a scalar field as

$$S_b = \int dxd\tilde{x} e^{-2d} \left(-P^{AB}\partial_A\varphi\partial_B\varphi - \frac{1}{2}m^2\varphi^2\right), \quad (7.4)$$

where φ is a real scalar field.

Now, we incorporate the KK ansatz and the assumption adopted in the previous section. In addition, we assume

$$\varphi(x, \tilde{x}) = \sum_{n=-\infty}^{\infty} \sum_{\tilde{n}=-\infty}^{\infty} \varphi_{n\tilde{n}}(x^i, \tilde{x}_i) e^{inx^1 + i\tilde{n}\tilde{x}_1}, \quad (7.5)$$

which satisfies $\varphi(x^1, \tilde{x}_1) = \varphi(x^1 + 2\pi, \tilde{x}_1 + 2\pi)$. Thus, n and \tilde{n} are integers. Then, the action becomes $\int dxd\tilde{x} \mathcal{L}_b$, where

$$\begin{aligned} \mathcal{L}_b \propto \sqrt{\hat{g}} e^{-2\phi} \sum_{n, \tilde{n}} \left[-\frac{1}{2} \hat{g}^{ij} \{(\partial_i - inA_i - i\tilde{n}B_i)\varphi_{n\tilde{n}}\}^* (\partial_j - inA_j - i\tilde{n}B_j)\varphi_{n\tilde{n}} \right. \\ \left. - \frac{1}{2} \left[m^2 + \left(\frac{1}{R}n + R\tilde{n} \right)^2 \right] |\varphi_{n\tilde{n}}|^2 \right]. \end{aligned} \quad (7.6)$$

Now, we find the KK tower of the mass spectrum, which is very similar to the left-mover mass operator in string theory with compactification [17]. In particular, there are infinite zero mode scalars in the case of $m = 0$ and $R = 1$. Note that we did not assume the condition for massless field in DFT, such as $\tilde{\partial}^a = 0$ or $\partial_a \tilde{\partial}^a = 0$.

The “level matching condition” $\partial_a \tilde{\partial}^a = 0$ was introduced into DFT to have the symmetry of the massless background field described by a novel algebra [4–9]. The massive scalar field introduced here originates only from the symmetry of the given fixed background fields such as in [12]. The validity of the so-called level matching condition on the field theory of the scalar is not yet clear.

If the condition $\partial_a \tilde{\partial}^a = 0$ or equivalently $\partial_A \partial^A = \eta^{AB} \partial_A \partial_B = 0$ is imposed on the action of the scalar field, the action can be expressed as

$$\begin{aligned} S_b &= \int dxd\tilde{x} e^{-2d} \left(-\frac{\eta^{AB} + \mathcal{H}^{AB}}{2} \partial_A \varphi \partial_B \varphi - \frac{1}{2} m^2 \varphi^2 \right) \\ \Rightarrow S_{\sharp} &= \int dxd\tilde{x} e^{-2d} \left(-\frac{1}{2} \mathcal{H}^{AB} \partial_A \varphi \partial_B \varphi - \frac{1}{2} m^2 \varphi^2 \right), \end{aligned} \quad (7.7)$$

which is the natural form for the action. It should be noted that this is possibly not the precise action, because the condition $\partial_a \tilde{\partial}^a = 0$ should be treated as a constraint on the field.

As for the KK mass spectrum discussed above, the condition becomes just $n\tilde{n} = 0$, and the mass spectrum can be derived from the action S_{\sharp} . Then, the spectrum is

$$m^2 + \frac{n^2}{R^2}, \quad m^2 + R^2 \tilde{n}^2, \quad (7.8)$$

and there is a single state for the zero mode if $m = 0$.

To briefly summarize the discussion thus far, the action S_{\flat} leads to a rather string-like spectrum for KK states but without level matching, whereas the action S_{\sharp} leads to the “physical” KK mass spectrum (with no stringy excitation mode).

Now, we consider two-dimensional toroidal compactification and the KK mass spectrum. The ansatz for the form of the scalar field is

$$\varphi(x, \tilde{x}) = \sum_{n_1, n_2, \tilde{n}_1, \tilde{n}_2} \varphi_{n_1 n_2 \tilde{n}_1 \tilde{n}_2}(x^i, \tilde{x}_i) e^{in_1 x^1 + in_2 x^2 + i\tilde{n}_1 \tilde{x}_1 + i\tilde{n}_2 \tilde{x}_2} \quad (7.9)$$

and for background geometry is

$$g_{\mu\nu} = \begin{pmatrix} R^2 & R^2 \tau_1 & 0 \\ R^2 \tau_1 & R^2 |\tau|^2 & 0 \\ 0 & 0 & \hat{g}_{ij} \end{pmatrix}, \quad (7.10)$$

$$g^{\mu\nu} = \begin{pmatrix} R^{-2} \tau_2^{-2} |\tau|^2 & -R^{-2} \tau_2^{-2} \tau_1 & 0 \\ -R^{-2} \tau_2^{-2} \tau_1 & R^{-2} \tau_2^{-2} & 0 \\ 0 & 0 & \hat{g}^{ij} \end{pmatrix}, \quad (7.11)$$

where we omit the vector degrees of freedom. For the antisymmetric field, we assume that only the non-zero element is $B \equiv b_{12}$.

We examine two cases. For the theory described by the action S_{\flat} , or without the condition $\partial_A \partial^A = 0$, the mass spectrum is found to be

$$m^2 + \left[\frac{1}{R\tau_2} \{n_2 + B\tilde{n}_1 - \tau_1(n_1 - B\tilde{n}_2)\} + R\tau_2 \tilde{n}_2 \right]^2 + \left[\frac{1}{R} (n_1 - B\tilde{n}_2) + R(\tilde{n}_1 + \tau_1 \tilde{n}_2) \right]^2, \quad (7.12)$$

as the sequence of mass squared. On the other hand, for the theory described by the action S_{\sharp} , or with the condition $\partial_A \partial^A = 0$, the mass spectrum is found to be

$$m^2 + \frac{1}{R^2 \tau_2^2} \left[\{n_2 + B\tilde{n}_1 - \tau_1(n_1 - B\tilde{n}_2)\}^2 + \tau_2^2 (n_1 - B\tilde{n}_2)^2 \right] + R^2 \left[(\tilde{n}_1 + \tau_1 \tilde{n}_2)^2 + \tau_2^2 \tilde{n}_2^2 \right], \quad (7.13)$$

with $n_1\tilde{n}_1 + n_2\tilde{n}_2 = 0$. The spectrum includes the following cases:

For $\tilde{n}_1 = \tilde{n}_2 = 0$,

$$m^2 + \frac{1}{R^2\tau_2^2} \left[(n_2 - \tau_1 n_1)^2 + \tau_2^2 n_1^2 \right]. \quad (7.14)$$

For $n_1 = n_2 = 0$,

$$m^2 + \left(\frac{B^2}{R^2\tau_2^2} + R^2 \right) \left[(\tilde{n}_1 + \tau_1 \tilde{n}_2)^2 + \tau_2^2 \tilde{n}_2^2 \right]. \quad (7.15)$$

For $\tilde{n}_1 = n_2 = 0$,

$$m^2 + \frac{|\tau|^2}{R^2\tau_2^2} (n_1 - B\tilde{n}_2)^2 + R^2 |\tau|^2 \tilde{n}_2^2. \quad (7.16)$$

For $n_1 = \tilde{n}_2 = 0$,

$$m^2 + \frac{1}{R^2\tau_2^2} (n_2 + B\tilde{n}_1)^2 + R^2 \tilde{n}_1^2. \quad (7.17)$$

In this section, we have examined the scalar field theory in the DFT background fields. Because of the duality in the background fields, the KK spectrum has a rich structure similar to that for the string theory with toroidal compactification.

VIII. A SIMPLY MODIFIED COSMOLOGICAL MODEL

Inspired by the success in the Hamiltonian analysis for the geodesic flow in the DFT background, we apply a similar method to a modified model for cosmology, which is related to the string cosmology or pre-big bang models [11] and, thus, also related to DFT.

In DFT, the field is not doubled but the coordinate appears to be doubled. The coordinate is of course not really doubled, since the level matching condition and $O(D, D)$ rotation leads to $\tilde{\partial}^a = 0$.

In the model here, we consider two metrics, g and \tilde{g} . Though our model describes a so-called bi-metric or bi-gravity theory, the degree of freedom is to be mildly restricted. The interesting points from a physical perspective are as follows. First, the model contains no inverse metric explicitly, which generally leads to an infinite series of perturbations in the metric. Second, we choose a model with no higher derivatives so as to abandon the perfect elimination of extra degrees of freedom. Third, we expect that the extra degrees of freedom in the metric can affect the resolution of cosmological problems.

For simplicity of the discussion, we consider $b_{\mu\nu} = 0$. The T-duality is expressed as a symmetry under $g_{\mu\nu} \rightarrow g^{\mu\nu}$. We wish to consider a new metric \tilde{g} and some moderate

constraint to obtain $\tilde{g}g = 1$ approximately. At the same time, we consider that $\tilde{g}g = 1$ does not hold strictly at the very beginning of our universe.

If the metric is really doubled,

$$P_{AB} \equiv \begin{pmatrix} \tilde{g} & 1 \\ 1 & g \end{pmatrix} \quad (8.1)$$

is not an exact projection matrix. Instead of adopting the strong constraint $P^2 = P$, we take

$$P\dot{\mathcal{H}}P = 0, \quad (8.2)$$

as a constraint where

$$\mathcal{H}_{AB} \equiv \begin{pmatrix} \tilde{g} & 0 \\ 0 & g \end{pmatrix}; \quad (8.3)$$

and dot denotes the derivative with respect to the canonical time.³

Of course, $P^2 = P$ implies $P\dot{\mathcal{H}}P = 0$, but the inverse does not always hold. Meanwhile, we expect the “relaxation” to $P^2 = P$ to depend on the dynamics.

To avoid treating infinite degrees of freedom and complicated canonical decomposition of the fields, we consider cosmological settings in the present paper. In other words, we consider a modification of the model shown in the last part of Sec. II. That is, we consider only the standard dynamics of finite degrees of freedom.

The action we first consider is

$$S \rightarrow - \int dt e^{-\Phi} \left[\frac{1}{8} \text{Tr} (\dot{M}\eta\dot{M}\eta) + \dot{\Phi}^2 + V \right], \quad (8.4)$$

where we add the constant potential V to the Lagrangian. We have an overall coefficient by the constant shift and redefinition of Φ . Then the action is rewritten as

$$S = -\frac{\lambda_s}{2} \int dt e^{-\Phi} \left[\frac{1}{8} \text{Tr} (\dot{M}\eta\dot{M}\eta) + \dot{\Phi}^2 + V \right], \quad (8.5)$$

where λ_s is the constant that represents the scale of string theory [11]. Further, we use a new time parameter with $dt = e^{-\Phi} d\tau$

$$S = -\frac{\lambda_s}{2} \int d\tau \left[\frac{1}{8} \text{Tr} (M'\eta M'\eta) + \Phi'^2 + e^{-2\Phi} V \right], \quad (8.6)$$

where the prime denotes differentiation with respect to τ .

³ Please remember that we are inspired by the canonical formalism for equations.

We now define the “pseudo”-projection matrices

$$P = \frac{\eta + M}{2}, \quad \bar{P} = \frac{\eta - M}{2}, \quad (8.7)$$

and we wish to enforce

$$PM'P = \bar{P}M'\bar{P} = 0, \quad (8.8)$$

using some constraints. Please note that thus far the “problem” is very similar to the case of the particle motion in the DFT background.

Now, the Lagrangian L_Λ with the constraint term is

$$L_\Lambda = \frac{\lambda_s}{2} \left[-\frac{1}{8} M'^{AB} M'_{AB} + \bar{\Lambda}_{AB} \bar{P}^{AC} M'_{CD} \bar{P}^{DB} + \Lambda_{AB} P^{AC} M'_{CD} P^{DB} - \Phi'^2 - e^{-2\Phi} V \right], \quad (8.9)$$

where A, B range over $1, \dots, D$. It must be noted that the constraint terms are invariant with respect to the choice of time in the action.

The conjugate momentum for M is derived as

$$\Pi^{AB} = \frac{\partial L_\Lambda}{\partial M'_{AB}} = \frac{\lambda_s}{2} \left[-\frac{1}{4} M'^{AB} + \bar{P}^{AC} \bar{\Lambda}_{CD} \bar{P}^{DB} + P^{AC} \Lambda_{CD} P^{DB} \right], \quad (8.10)$$

whereas the momentum for Φ is

$$\Pi_\Phi = \frac{\partial L_\Lambda}{\partial \Phi'} = -\lambda_s \Phi'. \quad (8.11)$$

Therefore, the Hamiltonian of the system becomes

$$\begin{aligned} H_\Lambda &= \Pi^{AB} M'_{AB} + \Pi_\Phi \Phi' - L_\Lambda \\ &= -\frac{4}{\lambda_s} \left[\Pi^{AB} - \frac{\lambda_s}{2} \left(\bar{P}^{AC} \bar{\Lambda}_{CD} \bar{P}^{DB} + P^{AC} \Lambda_{CD} P^{DB} \right) \right]^2 - \frac{1}{2\lambda_s} \Pi_\Phi^2 + \frac{\lambda_s}{2} e^{-2\Phi} V, \end{aligned} \quad (8.12)$$

where $(\mathcal{M}^{AB})^2$ means $\mathcal{M}^{AB} \mathcal{M}_{AB}$.

Finding an exact solution for the Lagrange multiplier is difficult or needs more recursive constraints. We consider simplification by using the assumed relation, $P^2 \simeq P$ and $\bar{P}^2 \simeq \bar{P}$. In this section, the symbol \simeq is used to indicate this assumed approximation adopted by us.

Then, we can express the following relations

$$\frac{\partial H_\Lambda}{\partial \bar{\Lambda}_{AB}} \simeq 2 \bar{P}^{AC} \left(\Pi_{CD} - \frac{\lambda_s}{2} \bar{\Lambda}_{CD} \right) \bar{P}^{DB} = 0, \quad (8.13)$$

and

$$\frac{\partial H_\Lambda}{\partial \Lambda_{AB}} \simeq 2 P^{AC} \left(\Pi_{CD} - \frac{\lambda_s}{2} \Lambda_{CD} \right) P^{DB} = 0. \quad (8.14)$$

The solutions of these equations are

$$\frac{\lambda_s}{2} \bar{\Lambda}_{AB} \simeq \Pi_{AB} + P_{AC} L_1^{CD} \bar{P}_{DB} + \bar{P}_{AC} L_2^{CD} P_{DB} + P_{AC} L_3^{CD} P_{DB}, \quad (8.15)$$

and

$$\frac{\lambda_s}{2} \Lambda_{AB} \simeq \Pi_{AB} + P_{AC} \bar{L}_1^{CD} \bar{P}_{DB} + \bar{P}_{AC} \bar{L}_2^{CD} P_{DB} + \bar{P}_{AC} \bar{L}_3^{CD} \bar{P}_{DB}, \quad (8.16)$$

where L_i and \bar{L}_i ($i = 1, 2, 3$) are arbitrary tensors.

Substituting these into the original Hamiltonian H_Λ , finally, we obtain the Hamiltonian in which the multipliers are eliminated as

$$\begin{aligned} H_\Lambda &\simeq -\frac{4}{\lambda_s} \left(\Pi^{AB} - \bar{P}^{AC} \Pi_{CD} \bar{P}^{DB} - P^{AC} \Pi_{CD} P^{DB} \right)^2 - \frac{1}{2\lambda_s} \Pi_\Phi^2 + \frac{\lambda_s}{2} e^{-2\Phi} V \\ &\simeq -\frac{8}{\lambda_s} \Pi^{AB} P_{BC} \Pi^{CD} \bar{P}_{DA} - \frac{1}{2\lambda_s} \Pi_\Phi^2 + \frac{\lambda_s}{2} e^{-2\Phi} V. \end{aligned} \quad (8.17)$$

We consider this as a new Hamiltonian

$$H_* \equiv -\frac{8}{\lambda_s} \Pi^{AB} P_{BC} \Pi^{CD} \bar{P}_{DA} - \frac{1}{2\lambda_s} \Pi_\Phi^2 + \frac{\lambda_s}{2} e^{-2\Phi} V. \quad (8.18)$$

The Hamilton equation thus becomes

$$M'_{AB} = \frac{\partial H_*}{\partial \Pi^{AB}} = -\frac{8}{\lambda_s} \left(\bar{P}_{AC} \Pi^{CD} P_{DB} + P_{AC} \Pi^{CD} \bar{P}_{DB} \right). \quad (8.19)$$

Therefore, $PM'P \simeq 0$ and $\bar{P}M'\bar{P} \simeq 0$ are verified.

The Wheeler-DeWitt equation can be obtained by the replacement

$$\Pi^{AB} \rightarrow \hat{\Pi}^{AB} = -i \frac{\delta}{\delta M_{AB}}, \quad \Pi_\Phi \rightarrow \hat{\Pi}_\Phi = -i \frac{\delta}{\delta \Phi}, \quad (8.20)$$

in the Hamiltonian and introducing the wave function Ψ satisfying

$$\hat{H}_* \Psi(M, \Phi) = 0, \quad (8.21)$$

modulo ordering of operators.

IX. “MINISUPERSPACE” VERSION OF THE MODIFIED MODEL

In this section, we examine the previous procedure of modification in the minisuperspace model. We suppose

$$M_{AB} = \begin{pmatrix} \tilde{A}(\tau) \delta^{ab} & 0 \\ 0 & A(\tau) \delta_{\mu\nu} \end{pmatrix}. \quad (9.1)$$

Since the kinetic part of M of the Lagrangian is

$$L_0 = -\frac{\lambda_s}{16} M'^{AB} M'_{AB} = -\frac{\lambda_s D}{8} A' \tilde{A}', \quad (9.2)$$

the conjugate momenta of A and \tilde{A} are

$$\pi \equiv \frac{\partial L_0}{\partial A'} = -\frac{\lambda_s D}{8} \tilde{A}', \quad \tilde{\pi} \equiv \frac{\partial L_0}{\partial \tilde{A}'} = -\frac{\lambda_s D}{8} A'. \quad (9.3)$$

Thus the naive kinetic part of the Hamiltonian becomes

$$H_0 = -\frac{8}{\lambda_s D} \pi \tilde{\pi} = -\frac{4}{\lambda_s D} (\pi \tilde{\pi} + \tilde{\pi} \pi). \quad (9.4)$$

To derive the modified model, it is useful to rewrite the above Hamiltonian as

$$H_0 = -\frac{4}{\lambda_s D} \text{Tr} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\pi} & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\pi} & 0 \\ 0 & \pi \end{pmatrix} \right]. \quad (9.5)$$

In this case, the modification explained in the previous section becomes

$$\begin{aligned} H_{*0} &= -\frac{2}{\lambda_s D} \text{Tr} \left[\begin{pmatrix} -\tilde{A} & 1 \\ 1 & -A \end{pmatrix} \begin{pmatrix} \tilde{\pi} & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} \tilde{A} & 1 \\ 1 & A \end{pmatrix} \begin{pmatrix} \tilde{\pi} & 0 \\ 0 & \pi \end{pmatrix} \right] \\ &= -\frac{2}{\lambda_s D} (\pi \tilde{\pi} + \tilde{\pi} \pi - A \pi A \pi - \tilde{A} \tilde{\pi} \tilde{A} \tilde{\pi}). \end{aligned} \quad (9.6)$$

The Hamiltonian for the minisuperspace version of our modified model is

$$H_* = -\frac{2}{\lambda_s D} (\pi \tilde{\pi} + \tilde{\pi} \pi - A \pi A \pi - \tilde{A} \tilde{\pi} \tilde{A} \tilde{\pi}) - \frac{1}{2\lambda_s} \Pi_\Phi^2 + \frac{\lambda_s}{2} e^{-2\Phi} V. \quad (9.7)$$

The Hamilton equations in the case with $V \equiv \text{constant}$ are found to be

$$A' = \frac{\partial H_*}{\partial \pi} = -\frac{4}{\lambda_s D} (\tilde{\pi} - A \pi A), \quad (9.8)$$

$$\tilde{A}' = \frac{\partial H_*}{\partial \tilde{\pi}} = -\frac{4}{\lambda_s D} (\pi - \tilde{A} \tilde{\pi} \tilde{A}), \quad (9.9)$$

$$\pi' = -\frac{\partial H_*}{\partial A} = -\frac{4}{\lambda_s D} \pi A \pi, \quad (9.10)$$

$$\tilde{\pi}' = -\frac{\partial H_*}{\partial \tilde{A}} = -\frac{4}{\lambda_s D} \tilde{\pi} \tilde{A} \tilde{\pi}, \quad (9.11)$$

and

$$\Phi' = \frac{\partial H_*}{\partial \Pi_\Phi} = -\frac{\Pi_\Phi}{\lambda_s}, \quad \Pi'_\Phi = -\frac{\partial H_*}{\partial \Phi} = \lambda_s V e^{-2\Phi}. \quad (9.12)$$

A special solution can be found for these equations. First, $\pi \equiv 0$ is found to be a solution for (9.10). Then, for (9.9) and (9.11), we find that $\tilde{A}\tilde{\pi} = -\frac{\lambda_s\sqrt{D}}{2}C = \text{constant}$ is a solution. Now, the solution is

$$\tilde{A}(\tau) = \frac{1}{A_0} \exp \left[-\frac{2}{\sqrt{D}}C(\tau - \tau_0) \right], \quad A(\tau) = A_0 \exp \left[\frac{2}{\sqrt{D}}C(\tau - \tau_0) \right] + \delta, \quad (9.13)$$

where A_0 , τ_0 , and δ are constants. Then, the solution for Φ is given by

$$\Phi(\tau) = C(\tau - \tau_0) \quad \text{for } V = 0, \quad (9.14)$$

$$\Phi(\tau) = \ln \left[\frac{\sqrt{V}}{C} \sinh C(\tau - \tau_0) \right] \quad \text{for } V > 0, \quad (9.15)$$

when the Hamiltonian constraint $H_* \approx 0$ is taken into consideration. The similarity to the known string cosmological solution [11] is obvious, up to the possible constant deviation δ in A . Note that here we use the “time” τ , and the original temporal parameter t is given by $t = \int e^{-\Phi} d\tau$. For readers’ convenience, we review the usual string-cosmological solution as a function of τ in Appendix A.

For the solution, we find that $A\tilde{A} \rightarrow 1$ when $\tau \rightarrow +\infty$. Of course, the solution exists for the case of time reversal as well. In order to determine the condition at which the deviation from $A\tilde{A} = 1$ is significant, some initial conditions must be considered. In the next section, we consider quantum cosmology of our model in the minisuperspace.

X. QUANTUM COSMOLOGY

Quantum cosmological treatment of the string cosmology has been widely studied [18]. In our model, we can obtain the minisuperspace Wheeler-DeWitt equation by replacing $\pi \rightarrow -i\frac{\partial}{\partial A}$, $\tilde{\pi} \rightarrow -i\frac{\partial}{\partial \tilde{A}}$, and $\Pi_\Phi \rightarrow -i\frac{\partial}{\partial \Phi}$ in (9.7). Thus, we obtain

$$\left[\frac{2}{\lambda_s D} \left(2 \frac{\partial}{\partial A} \frac{\partial}{\partial \tilde{A}} - A \frac{\partial}{\partial A} A \frac{\partial}{\partial A} - \tilde{A} \frac{\partial}{\partial \tilde{A}} \tilde{A} \frac{\partial}{\partial \tilde{A}} \right) + \frac{1}{2\lambda_s} \frac{\partial^2}{\partial \Phi^2} + \frac{\lambda_s}{2} e^{-2\Phi} V \right] \Psi = 0, \quad (10.1)$$

where Ψ is the wave function of the universe.⁴

Using the following variables, we can mostly simplify the above equation.

$$x = \frac{\sqrt{D}}{4} \ln A/\tilde{A}, \quad y = \frac{\sqrt{D}}{4} \ln A\tilde{A}. \quad (10.2)$$

⁴ In this toy model, we can easily neglect the operator ordering.

Up to the ordering, we have

$$\left[-\frac{1 + e^{-(4/\sqrt{D})y}}{2} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} \frac{1 - e^{-(4/\sqrt{D})y}}{2} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial \Phi^2} + \lambda_s^2 e^{-2\Phi} V \right] \Psi = 0. \quad (10.3)$$

Now, we consider the case with $V = \text{constant}$.

If we assume a solution with multiplicative form,

$$\psi(x, y, \Phi) = X(x)Y(y)Z(\Phi), \quad (10.4)$$

then the differential equation becomes

$$\begin{aligned} -\frac{1 + e^{-(4/\sqrt{D})y}}{2} \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} - \frac{1}{Y(y)} \frac{\partial}{\partial y} \frac{1 - e^{-(4/\sqrt{D})y}}{2} \frac{\partial Y(y)}{\partial y} \\ + \frac{1}{Z(\Phi)} \frac{\partial^2 Z(\Phi)}{\partial \Phi^2} + \lambda_s^2 e^{-2\Phi} V = 0 \end{aligned} \quad (10.5)$$

and can be separated as

$$-\frac{\partial^2 X_k(x)}{\partial x^2} = k^2 X_k(x), \quad -\frac{\partial^2 Z_K(\Phi)}{\partial \Phi^2} - \lambda_s^2 e^{-2\Phi} V Z_K(\Phi) - K^2 Z_K(\Phi) = 0, \quad (10.6)$$

and

$$-\frac{\partial}{\partial y} \frac{1 - e^{-(4/\sqrt{D})y}}{2} \frac{\partial Y_{kK}(y)}{\partial y} + \left(-K^2 + \frac{1 + e^{-(4/\sqrt{D})y}}{2} k^2 \right) Y_{kK}(y) = 0. \quad (10.7)$$

The non singular real solution of (10.7) at $y = 0$ can be found to be

$$Y_{kK}(y) = e^{-\sqrt{k^2 - 2K^2}y} F\left(\frac{1 + \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, \frac{1 - \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1; 1 - e^{-2y}\right), \quad (10.8)$$

where $F(\alpha, \beta, \gamma; z)$ is the Gauss' hypergeometric function.⁵ The solutions for (10.6) can be obtained from the standard quantum cosmological model reviewed in Appendix A.

If $K = \pm k$, $Y_{k\pm k}(y)$ has a maximum at $y = 0$. When we construct a wave packet for the cosmological wave function [18], the peak of this wave packet in terms of parameter y is naturally located at $y = 0$. This is because there is no other fixed peak apart from $y = 0$ in the constituent waves and waves destructively interfere with each other at $y \neq 0$. Thus, the approximate scale factor duality $A\tilde{A} \simeq 1$ is expected even at the “beginning” of the quantum universe. The detailed investigation on the behavior of the universe is left for future research.

⁵ Equivalent expressions for the solution are exhibited in Appendix B.

XI. SUMMARY AND OUTLOOK

The present paper consists of two parts. In the first part (Sec. III-VII), the motion of the particle in the background field in DFT has been investigated. We have explicitly shown that the geodesic in the $2D$ -dimensional doubled-spacetime cannot be the geodesic in the D -dimensional spacetime. The geodesic equation in the D -dimensional spacetime is derived by the Hamiltonian formalism of the constraint system and is found to be the geodesic flow equation. The KK excited states have been studied and are found to be the same spectra in string theory but without stringy excitations.

In the second part (Sec. VIII-X), we have considered the string cosmology with a bimetric model inspired by the constraint method discussed in the first part. Our method for the restriction on the metrics functions well, at least in the present reduced model for cosmology.

The canonical formalism for the model with two metrics should be further studied as the field theory of gravity. Of course, the important consideration is the deviation from the precise duality in the model. It will be interesting to find whether the extra degrees of freedom affects the cosmology and gravitational theory. We wish to apply our approach to other systems with some duality or other symmetries.

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**Appendix A: Classical and quantum equations in the standard string cosmology
(pre-big bang) model**

In this Appendix, we review the the standard string-cosmological solution [11, 18] using our notation for convenience. First, we define

$$M_{AB} = \begin{pmatrix} A^{-1}(\tau)\delta^{ab} & 0 \\ 0 & A(\tau)\delta_{\mu\nu} \end{pmatrix}, \quad (\text{A1})$$

where A is the square of the scale factor. Then, the action (8.6) is expressed as

$$S = -\frac{\lambda_s}{2} \int d\tau \left[-\frac{D}{4} \left(\frac{A'}{A} \right)^2 + \Phi'^2 + e^{-2\Phi} V \right] \equiv \int L d\tau. \quad (\text{A2})$$

The conjugate momentum for A and Φ is given by

$$\Pi \equiv \frac{\partial L}{\partial A'} = \frac{\lambda_s D}{4} \frac{A'}{A^2} \quad \text{and} \quad \Pi_\Phi \equiv -\lambda_s \Phi', \quad (\text{A3})$$

respectively. Now, we obtain the Hamiltonian as

$$H \equiv \Pi A' + \Pi_\Phi \Phi' - L = \frac{2}{\lambda_s D} A \Pi A \Pi - \frac{1}{2\lambda_s} \Pi_\Phi^2 + \frac{\lambda_s}{2} e^{-2\Phi} V. \quad (\text{A4})$$

For the case with $V = \text{constant}$, the Hamilton equations are

$$A' = \frac{\partial H}{\partial \Pi} = \frac{4}{\lambda_s D} A \Pi A, \quad \Pi' = -\frac{\partial H}{\partial A} = -\frac{4}{\lambda_s D} \Pi A \Pi, \quad (\text{A5})$$

$$\Phi' = \frac{\partial H}{\partial \Pi_\Phi} = -\frac{\Pi_\Phi}{\lambda_s}, \quad \Pi'_\Phi = -\frac{\partial H}{\partial \Phi} = \lambda_s V e^{-2\Phi}, \quad (\text{A6})$$

From (A5), we have

$$A \Pi = \frac{\lambda_s \sqrt{D}}{2} C, \quad (\text{A7})$$

and we find the solution

$$A(\tau) = A_0 \exp \left[\frac{2}{\sqrt{D}} C(\tau - \tau_0) \right], \quad (\text{A8})$$

where C , A_0 , and τ_0 are integration constants. Then, the solution for Φ is given by

$$\Phi(\tau) = C(\tau - \tau_0) \quad \text{for} \quad V = 0, \quad (\text{A9})$$

$$\Phi(\tau) = \ln \left[\frac{\sqrt{V}}{C} \sinh C(\tau - \tau_0) \right] \quad \text{for} \quad V > 0. \quad (\text{A10})$$

The Wheeler-DeWitt equation for the standard quantum string cosmology is [18]

$$\left[-\frac{2}{\lambda_s D} A \frac{\partial}{\partial A} A \frac{\partial}{\partial A} + \frac{1}{2\lambda_s} \frac{\partial^2}{\partial \Phi^2} + \frac{\lambda_s}{2} e^{-2\Phi} V \right] \Psi = 0. \quad (\text{A11})$$

By using the variable $x = \frac{\sqrt{D}}{2} \ln A$, we rewrite the equation as

$$\left[-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \Phi^2} + \lambda_s^2 e^{-2\Phi} V \right] \Psi(x, \Phi) = 0. \quad (\text{A12})$$

The general separable solution for a constant V is found to be

$$\left(\frac{\lambda_s \sqrt{V}}{2} \right)^{\pm ik} \Gamma(1 \mp ik) J_{\mp ik}(\lambda_s \sqrt{V} e^{-\Phi}) \xrightarrow{\Phi \rightarrow +\infty} e^{\pm ik\Phi - ikx}. \quad (\text{A13})$$

Appendix B: Other expressions for the solution (10.8)

The solution (10.8) can be expressed in the following forms:

$$\begin{aligned} Y_{kK}(y) &= e^{-\sqrt{k^2 - 2K^2}y} F\left(\frac{1 + \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, \frac{1 - \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1; 1 - e^{-2y}\right) \\ &= e^{+\sqrt{k^2 - 2K^2}y} F\left(\frac{1 + \sqrt{1 - k^2} - \sqrt{k^2 - 2K^2}}{2}, \frac{1 - \sqrt{1 - k^2} - \sqrt{k^2 - 2K^2}}{2}, 1; 1 - e^{-2y}\right) \\ &= e^{(1 - \sqrt{1 - k^2})y} F\left(\frac{1 - \sqrt{1 - k^2} - \sqrt{k^2 - 2K^2}}{2}, \frac{1 - \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1; 1 - e^{2y}\right) \\ &= e^{(1 + \sqrt{1 - k^2})y} F\left(\frac{1 + \sqrt{1 - k^2} - \sqrt{k^2 - 2K^2}}{2}, \frac{1 + \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1; 1 - e^{2y}\right) \\ &= e^{(1 - \sqrt{1 - k^2})y} F\left(\frac{1 - \sqrt{1 - k^2} - \sqrt{k^2 - 2K^2}}{2}, \frac{1 - \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1 - \sqrt{1 - k^2}; e^{2y}\right) \\ &= e^{(1 + \sqrt{1 - k^2})y} F\left(\frac{1 + \sqrt{1 - k^2} - \sqrt{k^2 - 2K^2}}{2}, \frac{1 + \sqrt{1 - k^2} + \sqrt{k^2 - 2K^2}}{2}, 1 + \sqrt{1 - k^2}; e^{2y}\right). \end{aligned} \quad (\text{B1})$$

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